REPORT ON THE GROUP ON SHIMURA VARIETIES WITHIN THE JUNIOR TRIMESTER PROGRAM ALGEBRA AND NUMBER THEORY

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Shimura varieties have recently received considerable attention in arithmetic geometry. Their cohomology plays an important role for the geometric realization of Langlands correspondences, a central topic in modern algebraic number theory. The group on Shimura varieties within the Junior trimester program "Algebra and number theory" united nine young researchers from all over the world who are working on the different aspects of this subject:

Xavier Caruso (IRMAR Université Rennes 1)
Elizabeth Csima (University of Illinois at Urbana-Champaign)
Qendrim Gashi (Max-Planck-Institut für Mathematik, Bonn)
Gerald Gotsbacher (Toronto)
Jérémy Le Borgne (IRMAR - Université Rennes 1)
Tuan Ngo Dac (Paris 13)
Brian Smithling (Toronto)
Eva Viehmann (Bonn)
Ying Zong (Toronto)

The HIM provided an environment which helped us to pursue our research projects, and to cooperate in small groups. We met about once a week for a specialised seminar. Here we presented our current work or closely related other new results in single talks or minicourses of 2-4 talks. The colloquium of the trimester program provided the opportunity for exchange with other participants working on related questions in the same area of mathematics. The close connection to the arithmetic geometry groups at the Mathematical Institute and the Max-Planck Institute for Mathematics in Bonn also contributed to an enspiring athmosphere for the program.

Let us briefly summarize some of the main topics of our work during the program.

Gotsbacher's interest is to study the cohomology of non-compact Shimura varieties, and thus also the cohomology and geometry of suitable compactifications. He gave a series of introductory talks on Mumford's canonical extension of automorphic vector bundles and on applications to compactifications and the cohomology of non-compact Shimura varieties.

Smithling worked on the topic of local models for Shimura varieties. He completed two research articles on topological flatness of different local models of Shimura varieties of PEL type. Furthermore he began to work on a survey article in the subject joint with Pappas and Rapoport. He met frequently with Rapoport while in Bonn as the work was progressing.

The reduction modulo p was the joint topic of interest for Csima, Gashi, and Viehmann. While in Bonn Gashi generalized his study of non-emptiness of affine Deligne-Lusztig varieties, and of Newton strata in the reduction modulo p of Shimura varieties. His method involves combinatorial questions on the so-called numbers game. He also discussed with Viehmann who came up with a different proof of these non-emptiness conditions using purity of Newton stratifications. Csima considered generalizations of filtrations and decompositions of so-called Hodge-Newton type for polarized p-divisible groups generalizing a classical theory of Katz.

The analogue of Shimura varieties over function fields are moduli spaces of Drinfeld shtukas with additional structure, so-called G-shtukas. In the same way as moduli spaces of abelian varieties are closely connected to moduli spaces of associated p-divisible groups, one also considers a local variant of G-shtukas. Contrary to Shimura varieties, the geometry of their moduli spaces is currently very little understood. During the program Ngo Dac and Viehmann investigated the relation between local and global G-shtukas in order to generalize the classical Serre-Tate theory for abelian varieties.

Generalizing in a different direction moduli spaces of p-divisible groups and of local G-shtukas also share combinatorial properties with Kisin varieties, a class of varieties recently defined by Pappas and Rapoport generalizing work of Kisin on deformations of Galois representations. During his stay in Bonn Caruso worked on his paper "Estimation des dimensions des variétés de Kisin". He discussed with Hellmann and Rapoport on Kisin varieties and with Viehmann on stratifications of affine Deligne-Lusztig varieties which he generalized to prove his new results.

The topic of Galois representations was also pursued in discussions of Caruso with Le Borgne (although they are both from Rennes, Caruso was spending a year at Moscow at the time) and of Le Borgne with Hellmann and Rapoport. Le Borgne's aim was to find an algorithm to compute the semi-simplification of a Galois representation mod p given by its (ϕ, Γ)-module. While in Bonn, he started to look into an analogue to the theory of characteristic polynomials for semilinear maps. A part of this work was later included in the preprint "Semi-characteristic polynomials, ϕ-modules and skew polynomials" (arxiv:1105.4083). Another part will appear in a forthcoming paper entitled "Un algorithme pour la réduction des ϕ-modules sur $k((u))$".

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Rational periodic points of rational maps of the projective line

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1 Introduction

Let φ be a morphism of degree 2 from the projective line to itself defined over $\mathbb{Q}$. When φ is a polynomial most is known about the existence of rational periodic points for this map. A conjecture states that in this case there are no rational periodic points of exact order N with $N > 3$. Moreover Poonen proved that there are no polynomials having both rational points of exact order 2 and rational points of exact order 3. If φ is any rational map of degree 2, one can easily find examples of rational periodic points of exact order N for $0 < N < 6$. It is interesting to study whether a rational map of degree 2 has both rational points of exact order N, rational points of exact order M and rational points of exact order H for $0 < N < 6$, $0 < M < 6$, $0 < H < 6$ and M, N, H three different natural numbers. This is the object of our present and future research.

2 Aim and scope

We are dealing with rational maps φ of $\mathbb{P}_1$ of degree 2. We want to study their rational periodic points, that is those points P such that $\varphi^N(P) = P$, where φ^N stands for φ iterated N times for some positive $N \in \mathbb{N}$. So let $f(X : Z), g(X : Z) \in \mathbb{Z}[X : Z]$ be homogeneous polynomials of degree 2, the object of our study are rational maps of the form

$$\varphi(X : Y) := \frac{f(X : Z)}{g(X : Z)}.$$

In order to study the problem in non-homogeneous form we simply set $f(x) := f(x : 1)$, $g(x) := g(x : 1)$ and

$$\varphi := \varphi(x) = \frac{f(x)}{g(x)}.$$

When $g(x) = 1$ it is known that there are no rational points of exact period 4 and 5 and it has been conjectured that the same holds for rational points of any period $N \geq 6$ (see [2]). In [1] Canci has provided a family of rational maps of $\mathbb{P}_1$ possessing a rational point of exact period 4. One can verify that every rational map of degree 2 with a rational periodic point of exact period 4 is equivalent to a rational map of the following form :

$$\varphi_4 = \varphi_{4,(c,d)} = \frac{(x-2)(4x-dc^2)}{2(x-c)(x-d)}$$

with some conditions on the integers c, d and with the 4-cycle given by $(0, c, \infty, 2)$. When $d = 1$ we recover the family given by Canci. In a similar way one can prove that every rational map of degree 2 with a rational periodic point of exact period 3 is equivalent to a rational map of this form:

$$\varphi_3 = \varphi_{3,(a,b)} = \frac{(x-1)(x-b)}{x(x-a)}$$

with some conditions on the integers a, b, and with the 3-cycle given by $(0, \infty, 1)$. Moreover one can prove that every rational family with a rational periodic point of exact period 5 can be written in the following form:

$$\varphi_5 = \varphi_{5,(c,d)} = \frac{2(x-d)(x-b)}{(x-c)(x-a)}$$

where

$$b = \frac{-c^3d + 4c^2d - 4c^2}{-cd^2 + 2d^2 + c^2d - 2c^2}$$

and

$$a = \frac{-2cd^2 + 8d^2 - 8d}{-cd^2 + 2d^2 + c^2d - 2c^2}$$

with some conditions on the integers c, d and with the 5-cycle given by $(0, c, \infty, 2, d)$. It has been conjectured that for $N \geq 6$ there is no rational map of degree 2 with a rational periodic point of exact period N. Here we would like to analyze, as Poonen did in [3] for polynomials, whether the families of rational maps φ_i $(i = 3, ..., 5)$ have other rational points of exact period $0 < N \leq 5$ and $0 < M \leq 5$ and $N \neq i \neq M \neq N$.

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On the Abelian Fundamental Group Scheme of a Family of Varieties

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1 Introduction

This paper has required several months of preparation and the last and definitive corrections have been added during my stay at H.I.M. where the article reached his final form before publication (cf. [2]). So let us first recall the most relevant results that will be described in the next section: let S be a connected Dedekind scheme and X an S-scheme provided with a section x. We prove that the morphism between fundamental group schemes $\pi_1(X,x)^{ab} \to \pi_1(\mathbf{Alb}_{X/S}, 0_{\mathbf{Alb}_{X/S}})$ induced by the canonical morphism from X to its Albanese scheme $\mathbf{Alb}_{X/S}$ (when the latter exists) fits in an exact sequence of group schemes $0 \to (\mathbf{NS}^{\tau}_{X/S})^{\vee} \to \pi_1(X,x)^{ab} \to \pi_1(\mathbf{Alb}_{X/S}, 0_{\mathbf{Alb}_{X/S}}) \to 0$ where the kernel is a finite and flat S-group scheme. Furthermore we prove that any finite and commutative quotient pointed torsor over the generic fiber X_η of X can be extended to a finite and commutative pointed torsor over X.

2 Aim and scope

A classical result states that the abelianized étale fundamental group of a complete smooth curve over a separably closed field is isomorphic to the étale fundamental group of its Jacobian (cf. [5], §9). In this paper we will generalize this result in the language of the fundamental group scheme introduced by Nori (cf. [6] and [7]) for schemes over fields then extended by Gasbarri (cf. [3]) for schemes over Dedekind schemes. In particular we prove that if $f : C \to S$ is a smooth and projective curve with integral geometric fibers endowed with a S-valued point $x \in C(S)$ then the natural morphism $\varphi : \pi_1(C,x)^{ab} \to \pi_1(J, 0_J)$ from the abelianized fundamental group scheme of C to the fundamental group scheme of its Jacobian is an isomorphism. This is a consequence of a more general statement in higher dimension already mentioned in the Introduction: if we replace C by a scheme of finite type X and J by $\mathbf{Alb}_{X/S}$, the Albanese scheme of X (provided it exists), then the morphism of fundamental group schemes $\pi_1(X,x)^{ab} \to \pi_1(\mathbf{Alb}_{X/S}, 0_{\mathbf{Alb}_{X/S}})$ induced by the canonical morphism $X \to \mathbf{Alb}_{X/S}$ fits in an exact sequence of group schemes

$0 \to (\mathbf{NS}^{\tau}_{X/S})^{\vee} \to \pi_1(X,x)^{ab} \to \pi_1(\mathbf{Alb}_{X/S}, 0_{\mathbf{Alb}_{X/S}}) \to 0$ where the kernel is the Cartier dual of a S-finite and smooth group scheme $\mathbf{NS}^{\tau}_{X/S}$, called torsion Néron-Severi scheme of X over S; when S is the spectrum of an algebraically closed field k of characteristic 0 this coincides with a classical statement (cf. [4], III, §4, Corollary 4.19 and [8], §5.8) whose main techniques are used here to solve our problem.

As already discussed in [1] the study of the fundamental group scheme is tightly related to the problem of extending finite and pointed torsors over the generic fiber X_η of X to torsors over X. In this paper we have concentrated our attention on commutative torsors. We already know that if X is an abelian scheme every (necessarily) commutative quotient (i.e. the group scheme acting on it is a quotient of the fundamental group scheme of X_η) pointed torsor over X_η can be extended to X (cf. [1], §3.2). Tossici recently proved in [9], Corollary 4.9 that if $S = Spec(R)$, where R is a d.v.r. of mixed characteristic and X is a normal scheme, faithfully flat over S with integral fibers then every commutative finite torsor (pointed or not) $Y' \to X_\eta$ with Y' connected such that the normalization Y of X in Y' has integral special fiber can be extended to a commutative finite torsor over X up to an extension of R. We have proved that for any Dedekind scheme S, under some existence assumptions on the schemes $\mathbf{Pic}^0_{X/S}$ and $\mathbf{Pic}^{\tau}_{X/S}$, every commutative finite pointed torsor $Y' \to X_\eta$ can be extended to a commutative finite pointed torsor over X. We do not need to extend the base scheme S and we do not need other assumptions on Y', but of course we may have problems while trying to solve existence problems for $\mathbf{Pic}^0_{X/S}$ and $\mathbf{Pic}^{\tau}_{X/S}$. However we also provide some examples where everything behaves well.

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